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(07) Find The Fourier Series for $f(x) = x+x^2$ in $[-\pi, \pi]$. Also show that Fourier Series which converge to $f(x)$ in $[-\pi, \pi]$, $f(x) = \pi^2$ when $x = \pm\pi$,

Soln. Given $f(x) = x + x^2$

we have $f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[x^2\right]_{-\pi}^{\pi} + \frac{1}{3\pi}\left[x^3\right]_{-\pi}^{\pi} = 0 + \frac{1}{3\pi}\left[\pi^3 + \pi^3\right]$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{=0} + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$g(x) = x\cos nx$

$g(-x) = -x\cos nx$

$= -g(x)$

$$= 0 + \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right]$$

$$= -\frac{4}{\pi n}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= +\frac{4}{\pi n^2}\left[x\cos nx\right]_0^{\pi} + \frac{4}{\pi n^2}\times 0$$

$$= \frac{4}{\cancel{\pi} n^2}\left[\cancel{\pi}(-1)^n\right] = \frac{4}{n^2}(-1)^n$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cdot\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cdot\sin nx\, dx + \int_{-\pi}^{\pi} x^2\cdot\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} x\cdot\sin nx\, dx + \int_{-\pi}^{\pi} x^2\cdot\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[2\left\{x(-)\frac{\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.(-)\frac{\cos nx}{n}dx\right]\right.$$

$$+ \frac{1}{\pi}\left[x^2(-)\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x(-)\frac{\cos nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx\right]$$

$$+ \frac{1}{\pi}\left[\cancel{-\frac{\pi^2}{n}\cos n\pi + \frac{\pi^2}{n}\cos(n\pi)} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\, dx\right]$$

$$= -\frac{2}{n}(-1)^n + 0 - \frac{2}{\pi n}\cdot\left[x\cdot\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}dx\right]$$

$$b_n = \frac{(-2)(-1)^n}{n} - \frac{2}{\pi n}\left[0 - 0\right]$$

$$\therefore f(x) = \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx\right) \text{ implies}$$

$$(x+x^2) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \Sigma\frac{(-2)(-1)^n}{n}\sin nx$$

$$(x+x^2) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos nx}{n^2} - \frac{\sin nx}{2n}\right]$$

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To show convergence of $f(x) = x + x^2$ at $x = \pm\pi$

We know

$$S(x) = \frac{1}{2}\left[f(\pi-0) + f(-\pi+0)\right]$$

$$= \frac{1}{2}\left[\cancel{\pi} + \pi^2 + (\cancel{-\pi}) + (-\pi)^2\right]$$

$$= \pi^2$$

i.e. $f(x) \longrightarrow \pi^2$

$\because f(x) = \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right) \Rightarrow$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos n\pi}{n^2} - \frac{\sin n\pi}{2n}\right]$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \cdot \frac{(-1)^n}{n^2} - 0$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \left[(-1)^{2n} = 1\right]$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Note: some authors writes $f(x)$ as on $[-\pi, \pi]$ or $[0, 2\pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

OR,

$$a_0 = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx$$

Q8. Expand $f(x)$ by Fourier-series where
$f(x) = e^x$ on $[-\pi, \pi]$

Solution. Given $f(x) = e^x$, $x \in [-\pi, \pi]$
We know Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -(1)$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{2}{\pi}\sinh\pi$$

$$\left[\because \sinh\pi = \left(\frac{e^{\pi} - e^{-\pi}}{2}\right)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

Formula $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

$$= \frac{1}{\pi}\left\{\frac{e^x}{1+n^2}\left[1.\cos nx + n\sin nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{\pi}(\cos n\pi + n\sin n\pi) - e^{-\pi}(\cos n\pi - n\sin n\pi)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{\pi}((-1)^n + 0) - e^{-\pi}(-1)^n\right\}$$

$(\because \sin n\pi = 0)$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)} \cdot 2\sinh\pi \qquad \left[\because \sinh\pi = \frac{e^{\pi} - e^{-\pi}}{2}\right]$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left\{e^x\left[1.\sin nx - n\cos nx\right]\right\}_{-\pi}^{\pi}$$

Formula $\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right]$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{\pi}(\sin n\pi - n\cos n\pi) - e^{-\pi}(-\sin n\pi - n\cos n\pi)\right]$$

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$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{\pi}(0-n\cos n\pi) - e^{-\pi}(0-n\cos n\pi)\right]$$

$$= \frac{n}{\pi(1+n^2)}\left[e^{-\pi} - e^{\pi}\right]\cos n\pi$$

$$= -\frac{n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]\cos n\pi$$

$$b_n = -\frac{n(-1)^n}{\pi(1+n^2)}(2\sinh\pi) \qquad \left[\because \frac{e^{\pi}-e^{-\pi}}{2} = \sinh\pi,\ \cos n\pi = (-1)^n\right]$$

$$\therefore f(x) = \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right) \Rightarrow$$

$$e^x = \frac{1}{2}\cdot\left(\frac{2}{\pi}\sinh\pi\right) + \sum_{n=1}^{\infty}\frac{(-1)^n \cdot 2\sinh\pi}{\pi(1+n^2)}\cos nx$$

$$+ \sum_{n=1}^{\infty}\frac{n(-1)^n(-)}{\pi(1+n^2)}(2\sinh\pi)\sin nx$$

$$\Rightarrow e^x = \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\frac{2(-1)^n\sinh\pi}{\pi(1+n^2)}\left[\cos nx - n\sin nx\right]$$

Required expansion.

Expand $f(x) = \sin x$, $0 < x < \pi$ in a Fourier series.

Also deduce that $\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$.

Solution. Given $f(x) = \sin x$, $x \in]0, \pi[$

We know by Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$, \quad b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Now,

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$a_0 = -\frac{1}{\pi}\left[\cos\pi - 1\right] = -\frac{1}{\pi}\left[-1-1\right] = \frac{2}{\pi}$$

$$\Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$\therefore a_n = \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$a_n = \frac{1}{2\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{(n-1)}\right\} - \left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right]$$

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$$a_n = \frac{1}{2\pi}\left[\left(\frac{(-)(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-n+1+n+1}{n^2-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n(-1)^{-1}}{n-1}\right) - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{n-1-n-1}{n^2-1}\right) - \frac{2}{n^2-1}\right]$$

$$a_n = \frac{1}{2\pi}\left[\frac{(-1)^n(-2)}{n^2-1} - \frac{2}{n^2-1}\right] \quad n \geq 2$$

$$a_n = \frac{-2}{2\pi}\left[\frac{(-1)^n+1}{(n^2-1)}\right], \quad n \geq 2$$

$$a_n = -\frac{1}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right], \quad n \geq 2$$

For $n=1$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = 0 \quad \text{clearly}$$

$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \Rightarrow$

$$\sin x = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi} + 0 + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots$$

$$= \frac{1}{\pi} + \left[\left(-\frac{1}{\pi}\right)\frac{2}{3}\cos 2x + \left(-\frac{1}{\pi}\right)0 + \left(-\frac{1}{\pi}\right)\frac{2}{15}\cos 4x + \cdots\right]$$

$$\sin x = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \cdots\right]$$

Put $x=0$, we get

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \quad \text{Proved.}$$

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Problems for Practice

(1) Expand $f(x) = x$, $x \in]0, 2\pi[$ by Fourier series
Ans: $x = \pi - 2[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots]$

(2) Expand $f(x)$ by Fourier series where

(i) $f(x) = \pi - x$, $0 < x < 2\pi$

Ans. $\pi - x = 2[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots]$

(ii) $f(x) = x \sin x$, $x \in]0, 2\pi[$

Ans: $x\sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \dots\right]$

(3) Find Fourier series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Ans. $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$
$+ \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$

For discontinuous function, Fourier series gives the value of $f(x)$ as the arithmatic mean of left hand limit and right hand limit

i.e. At the point of discontinuity at $x = c$

$$f(x) = \frac{1}{2}[\text{LHL of } f(x) \text{ at } x=c + \text{RHL of } f(x) \text{ at } x=c]$$

$$f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$$

Example. Find Fourier series for $f(x)$ where

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hint.
Here $f(x)$ is discontinuous at $x=0$

So $f(x) = \frac{1}{2}[\text{LHL of } f(x) \text{ at } x=0 + \text{RHL of } f(x) \text{ at } x=0]$

$f(x) = \frac{1}{2}[f(0-0) + f(0+0)]$ [Here $c=0$]

Ans: $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Any way Let us solve.....

We know Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -[x]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= -[0-(-\pi)] + \frac{1}{2\pi}[\pi^2 - 0^2]$$

$$a_0 = -\pi + \frac{\pi}{2} \Rightarrow \boxed{a_0 = -\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$a_n = \frac{1}{\pi}(-)\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\frac{\sin nx}{n}\,dx\right]$$

$$= 0 + \frac{1}{\pi}[0] + \frac{1}{\pi n}\cdot\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$\boxed{a_n = +\frac{1}{\pi n^2}[\cos n\pi - 1]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$b_n = \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} 1\cdot\frac{\cos nx}{n}\,dx\right]$$

$$b_n = \frac{1}{n}\left[1-\cos n\pi\right] - \frac{1}{\pi n}\left[\pi\cos n\pi\right] + \frac{1}{\pi n}\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + 0$$

$$b_n = \frac{1-2\cos n\pi}{n}$$

$$\therefore f(x) = -\frac{\pi}{4} + \sum \frac{1}{\pi n^2}(\cos n\pi - 1)\cos nx$$
$$+ \sum \left(\frac{1-2\cos 2n\pi}{n}\right)\sin nx$$

but $x = 0$

$$f(0) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1) + 0$$

$$= -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$f(0) = -\frac{\pi}{4} + \left(\frac{-2}{\pi}\right)\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \quad \text{—(1)} \qquad \left[\because \{(-1)^n - 1\} = 0, \text{ if } n \text{ is even}\right]$$

Now, $f(x)$ is discontinuous at $x=0$

and $f(0-0) = -\pi \qquad \left[\because \lim_{h\to 0} f(0-h) = -\pi\right]$

$$f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} h = 0$$

$$\therefore f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right]$$
$$= \frac{1}{2}\left[\lim_{h\to 0} f(0-h) + \lim_{h\to 0} f(0+h)\right]$$

$$f(0) = \frac{1}{2}\left[-\pi + 0\right] = -\frac{\pi}{2} \quad \text{—(2)}$$

$\therefore$ Eqn (1) and (2) $\Rightarrow$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right].$$

or, $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

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(4) Find the Fourier series expansion of the periodic function of period 2π defined as

$$f(x) = \begin{cases} x, & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

Ans. $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$.

(5) Find the Fourier series of the function

$$f(x) = \begin{cases} x+\pi, & \text{for } 0 \le x \le \pi \\ -x-\pi, & \text{for } -\pi \le x < 0 \end{cases}$$

Ans. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right]$

(6) Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Ans. $f(x) = \frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$.

(7) Find the Fourier series of the function

$$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0 \\ \frac{\pi}{4}, & 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0$

Ans. $f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots$.

(8) Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

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(9) Find Fourier of the following functions:

(i) $$f(x) = \begin{cases} 0 & , 0 < x < \pi \\ 1 & , \pi < x < 2\pi \end{cases}$$

Ans: $f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

(ii) $$f(x) = \begin{cases} 1 & , 0 < x < \pi \\ 2 & , \pi < x < 2\pi \end{cases}$$

Ans. $f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

(iii) $$f(x) = \begin{cases} 0 & , -\pi < x \le 0 \\ \frac{\pi x}{4} & , 0 < x < \pi \end{cases}$$

Ans. $f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx\right]$

(iv) $$f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \\ -\frac{\pi}{2} & \text{for } x = 0 \end{cases}$$

Ans $f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 3\sin x - \frac{1}{2}\sin 2x + \cdots$

(v) $$f(x) = \begin{cases} \pi + x & , -\pi < x < -\frac{\pi}{2} \\ \frac{\pi}{2} & , -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & , \frac{\pi}{2} < x < \pi \end{cases}$$

Ans: $f(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$

Fourier expansion of an even function.

Let $f(x)$ be an even function on $[-\pi, \pi]$,

then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

Note. Here $b_n = 0$.

Example. Find the Fourier series expansion of $f(x)$ where

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence prove that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Ans. See Q. 5. (ii) part.

Fourier expansion of an odd function

Let $f(x)$ be an odd function on $[-\pi, \pi]$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

as $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$

, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

Example. Find Fourier series expansion of $f(x) = x^3$ on $]-\pi, \pi[$.

Soln. Clearly $f(x) = x^3$ is an odd function as $f(-x) = -f(x)$.

$\therefore a_0 = 0$ and $a_n = 0$

$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\,dx$

We know $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ etc, $v_1 = \int v\,dx$, $v_2 = \int v_1\,dx$ etc

So,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin nx \, dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x) = -f(x)\right]$$

$$b_n = \frac{2}{\pi}\left[x^3(-)\frac{\cos nx}{n} - 3x^2 \cdot \left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{-\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^3 \cos n\pi}{n} + \frac{3\pi^2}{n^2}\sin n\pi + \frac{6\pi}{n^3}\cos n\pi - \frac{6}{n^4}\sin n\pi + 0 - 0 - 0 + 0\right]$$

$$b_n = \frac{-2\pi^2 \cos n\pi}{n} + 0 + \frac{2}{\pi}\left(\frac{6\pi}{n^3}\right)\cos n\pi$$

$$= \frac{-2\pi^2(-1)^n}{n} + \frac{12}{n^3}(-1)^n$$

$$b_n = 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \Rightarrow$$

$$x^3 = \sum_{n=1}^{\infty} 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]\sin nx$$